

## UNSYMMETRICAL BENDING AND SHEAR CENTRE

This means that the plane of loading is parallel to the plane containing the principal centroidal axis of the inertia of cross-section of the beam. This type of bending is known as symmetrical bending.

### PROPERTIES OF BEAM CROSS-SECTION

The integral  $\int xy \, dA$  is known as product of inertia and the pair of axes, for which it is zero, are known as principal axes of the cross-section.

### PRINCIPAL MOMENTS OF INERTIA:-

The principal axes of any area are those axes about which the product of inertia ( $I_{xy}$ ) is zero. Axes of symmetry through centroid are automatically principal axes, as the moments for opposite quadrants cancelling each other out.

If  $Ox$  and  $Oy$  are two perpendicular axes through centroid and  $Ou$  and  $ov$  are principal axes at an angle  $\theta$  with axes  $Ox$  and  $Oy$  in anticlockwise direction.

(i) Condition for principal axes is,

$$\tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}}$$

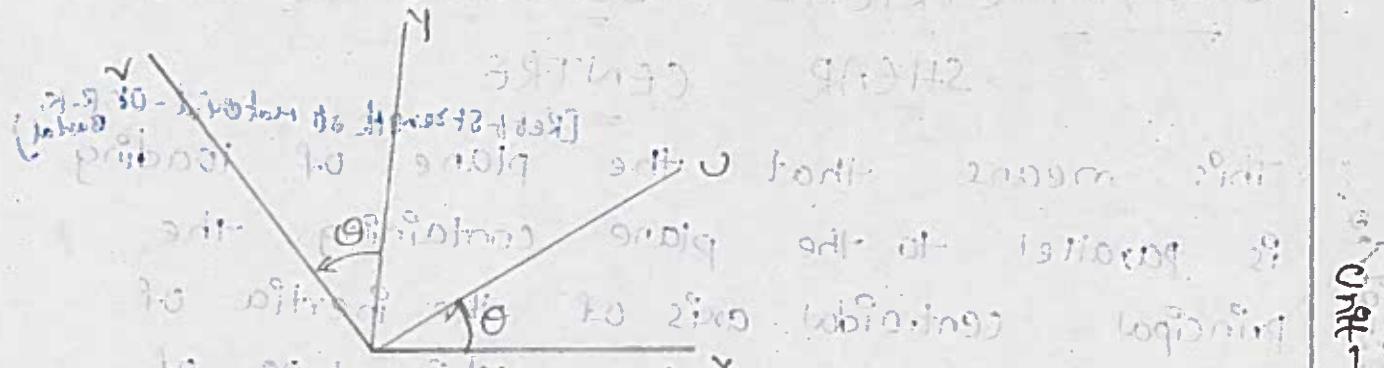
(ii) Principal moments of inertia about axes  $Ou$  and  $ov$  are:

$$I_{uu} = \frac{1}{2} (I_{xx} + I_{yy}) + \frac{1}{2} (I_{xx} - I_{yy}) \cos 2\theta - I_{xy} \sin 2\theta$$

$$I_{vv} = \frac{1}{2} (I_{xx} + I_{yy}) + \frac{1}{2} (I_{xx} - I_{yy}) \sec 2\theta$$

$$I_{vv} = \frac{1}{2} (I_{xx} + I_{yy}) - \frac{1}{2} (I_{xx} - I_{yy}) \sec 2\theta$$

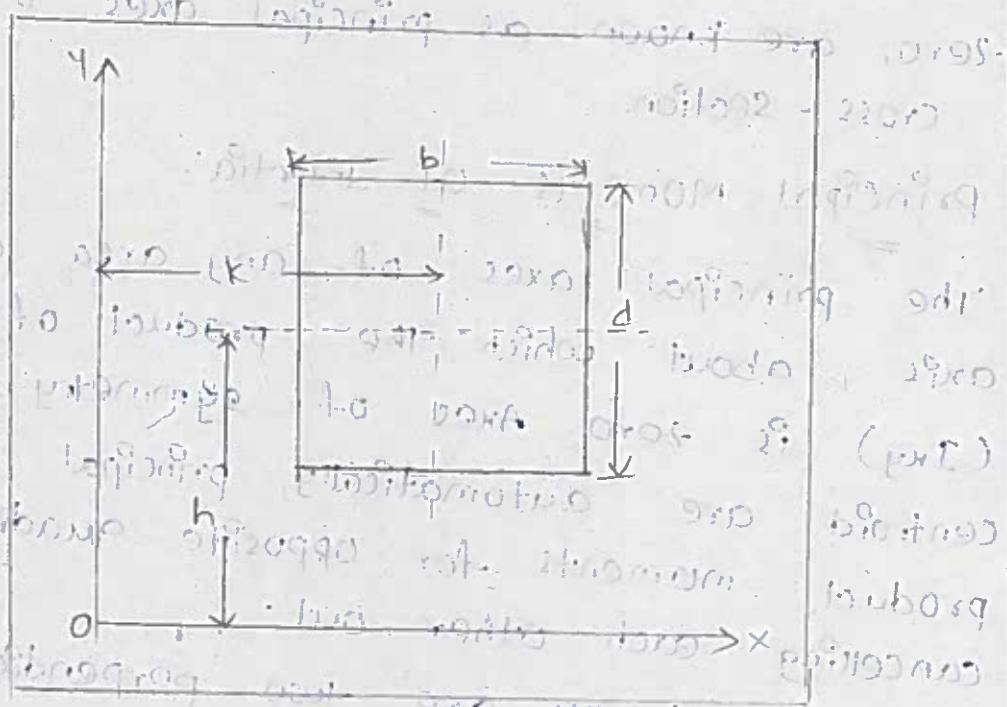
ALSO  $I_{uu} + I_{vv} = I_{xx} + I_{yy}$



(iii) The co-ordinates  $u, v$  relative to  $0u, 0v$  and  $x, y$  relative to  $ox, oy$  of any point  $p$  are given by

$$u = x \cos \theta + y \sin \theta \text{ and } v = y \cos \theta - x \sin \theta$$

$$v = y \cos \theta - x \sin \theta$$



(iv) If  $I_{xx}$  and  $I_{yy}$  are equal, then  $\theta$  will be  $90^\circ$  and  $\cos 2\theta = 0$ . And  $\sec 2\theta$

$$= \frac{1}{\cos 2\theta} = \frac{1}{0} = \infty.$$

(v) A rectangle of width  $b$  and depth  $d$ , the sides of the rectangle are parallel to the principal axes. The product of inertia  $I_{xy}$  will be 0.

$$I_{xy} = \int xy \, dA = \iint xy \, dy \, dx = \left( \frac{x^2}{2} \right) \Big|_{k-\frac{b}{2}}^{k+\frac{b}{2}} \times \left( \frac{y^2}{2} \right) \Big|_{h-\frac{d}{2}}^{h+\frac{d}{2}}$$

$$= \frac{1}{2} \left( k^2 - \left( k - \frac{b}{2} \right)^2 \right) \frac{1}{2} + \left( k^2 + \left( k + \frac{b}{2} \right)^2 \right) \frac{1}{2} = \frac{b^2 d^2}{12}$$

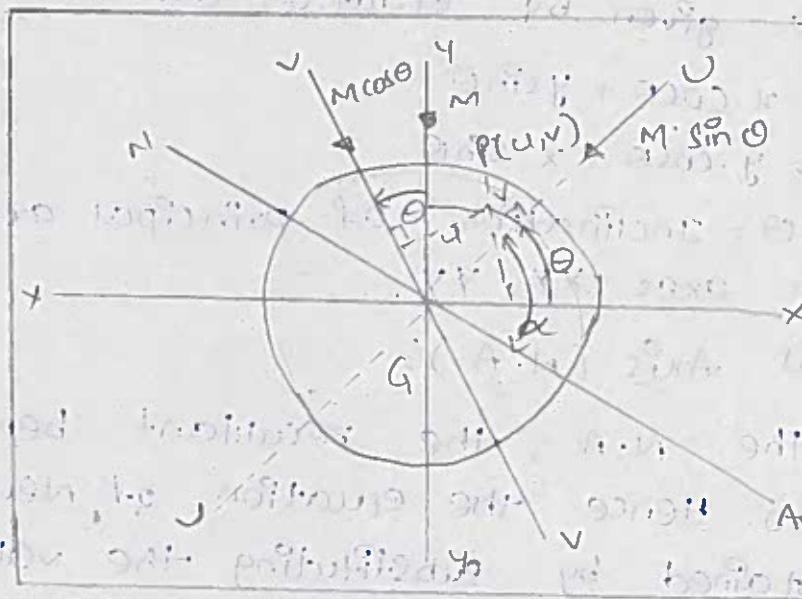
$$= \left[ \frac{\left(\frac{k+b}{2}\right)^2 - \left(\frac{k-b}{2}\right)^2}{2} \right] \times \left[ \frac{\left(\frac{h+d}{2}\right)^2 - \left(\frac{h-d}{2}\right)^2}{2} \right]$$

$$\Rightarrow kb \times hd = bd \times hk \quad (\because b \times d = A)$$

Hence the product of inertia of a rectangle whose sides are parallel to the axes is equal to area of rectangle  $\times$  distance of its C.G from one axis  $\times$  distance of its C.G from other axis.

### Stress in unsymmetrical Bending

The cross-section of a beam subjected to a bending moment  $M$  in the plane of  $Y-Y$  is shown in fig: the co-ordinate axes  $xx$  and  $yy$  pass through the centroid  $G$  of the section.



Let  $uv, vv$  = principal axes passing through  $G$  and inclined at the angle  $\theta$  to  $xx$  and  $yy$  axes respectively.

the moment in plane  $uv = M \sin \theta$

Moment in plane  $vv = M \cos \theta$

The moment in plane  $uv$ , will bend the beam about axis  $vv$ . The bending stress ( $\sigma_b$ ) due to this moment will be equal to

$$\frac{(M \sin \theta) \times 4}{I_{vv}} \quad \left[ \frac{M}{I} = \frac{\sigma}{y} \text{ (cor)} \right] \sigma = \frac{M}{I} \times y$$

Similarly; the moment in plane  $vv$ , will bend the beam about axis  $vv$ . The bending stress due to this moment will be equal to  $(M \cos \theta) \times v$ .

Hence, the resultant bending stress at any point  $P(u, v)$  will be given by

$$\sigma_b = \frac{(M \sin \theta) \times u}{I_{vv}} + \frac{(M \cos \theta) \times v}{I_{vv}}$$

$$= M \left[ \frac{u \sin \theta}{I_{vv}} + \frac{v \cos \theta}{I_{vv}} \right]$$

In the above equation, the signs of  $u$  and  $v$  will determine the nature of bending stress. If the co-ordinates of a point with respect to  $xx$ ,  $yy$  axes are known then the co-ordinates of the same point with respect of  $vv$ ,  $vv$  axes will be given by equation as.

$$u = x \cos \theta + y \sin \theta$$

$$v = y \cos \theta - x \sin \theta$$

where  $\theta$  = Inclination of principal axes  $vv, vv$  with axes  $xx, yy$ .

Neutral Axis (N.A.):

At the N.A., the resultant bending stress is zero. Hence the equation of Neutral axis is obtained by substituting the value of  $\sigma_b$  equal to zero in eqn.

Hence for neutral axis,

$$M \left[ \frac{u \sin \theta}{I_{vv}} + \frac{v \cos \theta}{I_{vv}} \right] = 0$$

$$\frac{u \sin \theta}{I_{vv}} + \frac{v \cos \theta}{I_{vv}} = 0$$

$$\frac{v \cos \theta}{I_{vv}} = -\frac{u \sin \theta}{I_{vv}}$$

$$v = -\frac{u \sin \theta}{I_{vv}} \times \frac{I_{vv}}{\cos \theta}$$

$$= -\left[ \frac{I_{vv}}{I_{vv}} \times \frac{\sin \theta}{\cos \theta} \right] u = \left( -\frac{I_{vv}}{\cos \theta} \tan \theta \right) u$$

is the equation of a straight line  
 $(y = mx + c)$  passing through the centroid  $G$   
 of the section. Here  $m = \left( -\frac{I_{UU} \tan \theta}{I_{VV}} \right)$  is the slope  
 of neutral axis.

Slope of Neutral Axis:

Let  $\alpha$  = Angle made by neutral axis with  
 axis  $UU$ .

Then  $\tan \alpha$  = slope of the neutral axis.

The slope of neutral axis is given as

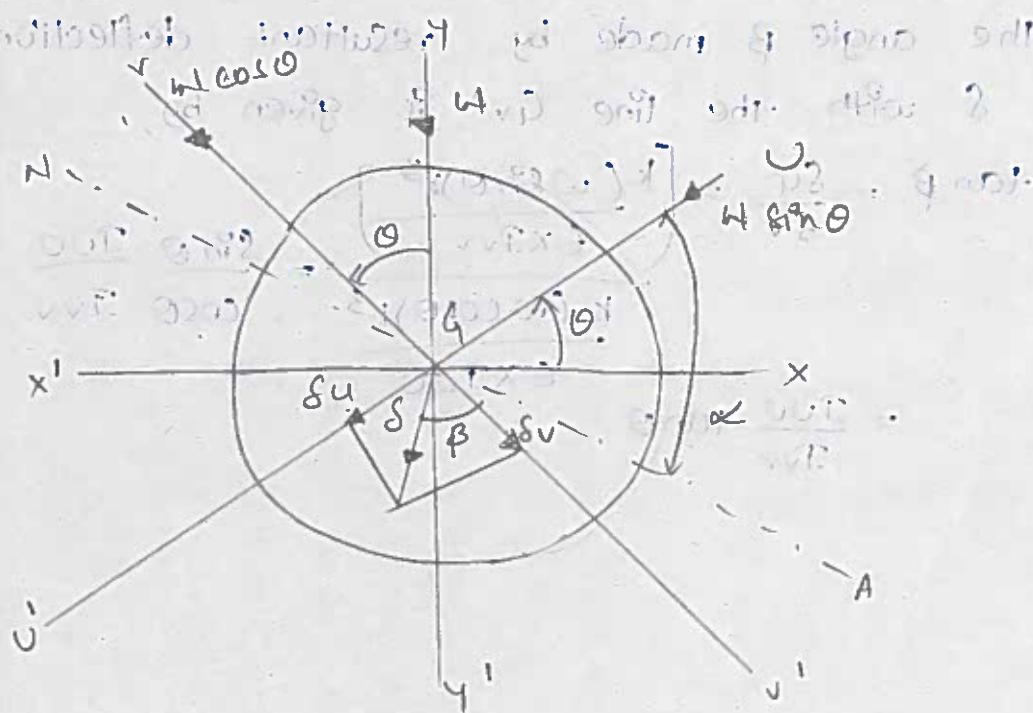
$$\tan \alpha = m = \left[ -\frac{I_{UU} \tan \theta}{I_{VV}} \right]$$

$$\alpha = \tan^{-1} \left[ -\frac{I_{UU} \tan \theta}{I_{VV}} \right]$$

Deflection of Beams in Unsymmetrical

Bending:

A transverse section of a beam with centroid  $G$  is shown in fig. along with rectangular co-ordinate axes  $xx'$  and  $yy'$ . The principal axes  $UU'$  and  $VV'$  inclined at an angle  $\theta$  to the  $xy$  set of co-ordinate axes are also shown.  $w$  is the load acting along line  $YY'$ . This load can be resolved into two components load  $w \sin \theta$  along  $UG$  and  $w \cos \theta$  along  $VG$ .



The component  $w \sin \theta$ , will bend the beam about  $VV'$  axis whereas,  $w \cos \theta$  will bend the beam about  $UU'$  axis.

Let  $\delta_u$  = deflection due to load  $w \sin \theta$  along line  $UU'$  and

$\delta_v$  = deflection due to load  $w \cos \theta$  along line  $VV'$

The values of  $\delta_u$  and  $\delta_v$  are given as,

$$\delta_u = \frac{k(w \sin \theta) L^3}{E \times I_{VV}} \text{ and}$$

$$\delta_v = \frac{k(w \cos \theta) L^3}{E \times I_{UU}}$$

• The resultant deflection ' $\delta$ ' is obtained as

$$\delta = \sqrt{\delta_u^2 + \delta_v^2}$$

$$= \sqrt{\left[ \frac{k(w \sin \theta) L^3}{E \times I_{VV}} \right]^2 + \left[ \frac{k(w \cos \theta) L^3}{E \times I_{UU}} \right]^2}$$

$$= \frac{k w L^3}{E} \sqrt{\frac{\sin^2 \theta}{I_{VV}} + \frac{\cos^2 \theta}{I_{UU}}}$$

Note:  $K = \frac{1}{48}$  for a c.s. beam carrying a point load at the centre.

The angle  $\beta$  made by resultant deflection  $\delta$  with the line  $VV'$  is given by,

$$\tan \beta = \frac{\delta_u}{\delta_v} = \frac{\frac{k(w \sin \theta) L^3}{E \times I_{VV}}}{\frac{k(w \cos \theta) L^3}{E \times I_{UU}}} = \frac{\sin \theta}{\cos \theta} \frac{I_{UU}}{I_{VV}}$$

$$= \frac{I_{UU}}{I_{VV}} \tan \theta$$

Problem :- Showc an unequal angle of dimension 100 mm by 60 mm and 10 mm thick.

Determine:

- Position of the principal axes and
- Magnitude of the principal moments of inertia for the given angle.

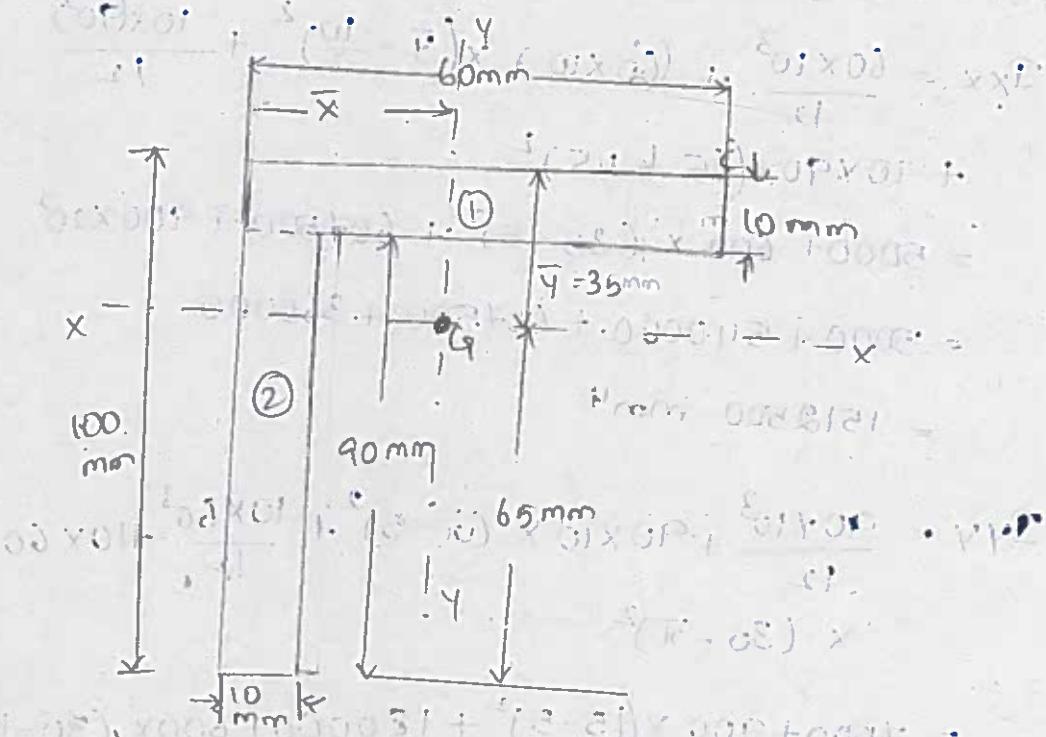
Sol:- first find the location of centroid G of the given angle.

Divide the angle into two rectangles 1 and 2.

then

$$A_1 = 60 \times 10 = 600 \text{ mm}^2$$

$$A_2 = (100 - 10) \times 10 = 90 \times 10 = 900 \text{ mm}^2$$



Take horizontal distances from line AB and vertical distances from line AC vertically downwards. Let  $(x_1, y_1)$  and  $(x_2, y_2)$  and the co-ordinates of the C.G. of rectangles (1) and (2).

$$x_1 = \frac{60}{2} = 30 \text{ mm}, y_1 = \frac{10}{2} = 5 \text{ mm}$$

$$x_2 = \frac{10}{2} = 5 \text{ mm}, y_2 = 10 + \frac{90}{2} = 55 \text{ mm}$$

Now the centroid G, is obtained as

$$\bar{x} = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2} = \frac{600 \times 3 + 900 \times 5}{600 + 900} = \frac{1800 + 4500}{1500}$$

$$= 12 + 3 = 15 \text{ mm and}$$

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{600 \times 5 + 900 \times 55}{600 + 900} = \frac{3000 + 49500}{1500}$$

$$= 2 + 33 = 35 \text{ mm}$$

Hence the centroid G is below line AC at a distance of  $\bar{y} = 35 \text{ mm}$  and towards right of line AB at a distance of  $\bar{x} = 15 \text{ mm}$

Draw a horizontal line x-x and vertical line y-y through G. Then x-x and y-y are the axes through centroid.

Now the moments of inertia about x-axis, about y-axis and product of inertia of the given angle are obtained as

$$I_{xx} = \frac{60 \times 10^3}{12} + (60 \times 10) \times \left(5 - \frac{10}{2}\right)^2 + \frac{10 \times 90^3}{12}$$

$$+ 10 \times 90 \times (65 - 45)^2$$

$$= 5000 + 600 \times (35 - 5)^2 + 607500 + 900 \times 20^2$$

$$= 5000 + 540000 + 607500 + 360000$$

$$= 1512500 \text{ mm}^4$$

$$I_{yy} = \frac{90 \times 10^3}{12} + 90 \times 10 \times (\bar{x} - 5)^2 + \frac{10 \times 60^3}{12} + 10 \times 60$$

$$\times (30 - \bar{x})^2$$

$$= 7500 + 900 \times (15 - 5)^2 + 180000 + 600 \times (30 - 15)^2$$

$$= 7500 + 90000 + 180000 + 600 \times 225$$

$$= 7500 + 90000 + 180000 + 135000 = 412500 \text{ mm}^4$$

and

As the sides of the rectangles 1 and 2 are parallel to axes x-x and y-y, hence product of inertia is given by equation

$$k_1 = (30 - \bar{x}) = 30 - 15 = 15 \text{ mm}$$

$$h_1 = (\bar{y} - 5) = (35 - 5) = 30 \text{ mm}$$

$k_2 = (\bar{x} - 5) = 15 - 5 = 10 \text{ mm}$ . It is towards left of Y-Y. Hence -ve.

$$= -10 \text{ mm}$$

$h_2 = (65 - 45) = 20 \text{ mm}$ . It is below X-X and hence is -ve.

$$= -20 \text{ mm}$$

$$( \because A_1 = 600$$

$$\underline{A_2 = 900})$$

$$I_{xy} = 600 \times 15 \times 30 + 900 \times (-10) (-20)$$

$$= 270000 + 180000 = 450000 \text{ mm}^4$$

i) Position of principal axes.

for principal axes,

$$\tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}}$$

$$= \frac{2 \times 450000}{412500 - 1512500} = -0.818$$

$\tan 2\theta$  is -ve in 2nd quadrant

$$2\theta = \tan^{-1}(-0.818) = -39.28^\circ = 180 - 39.28^\circ$$

$$\theta = \frac{140.72^\circ}{2} = 70.36^\circ$$

(ii) value of principal moment of inertia  
moment of inertia about principal axes

O-O is given by,

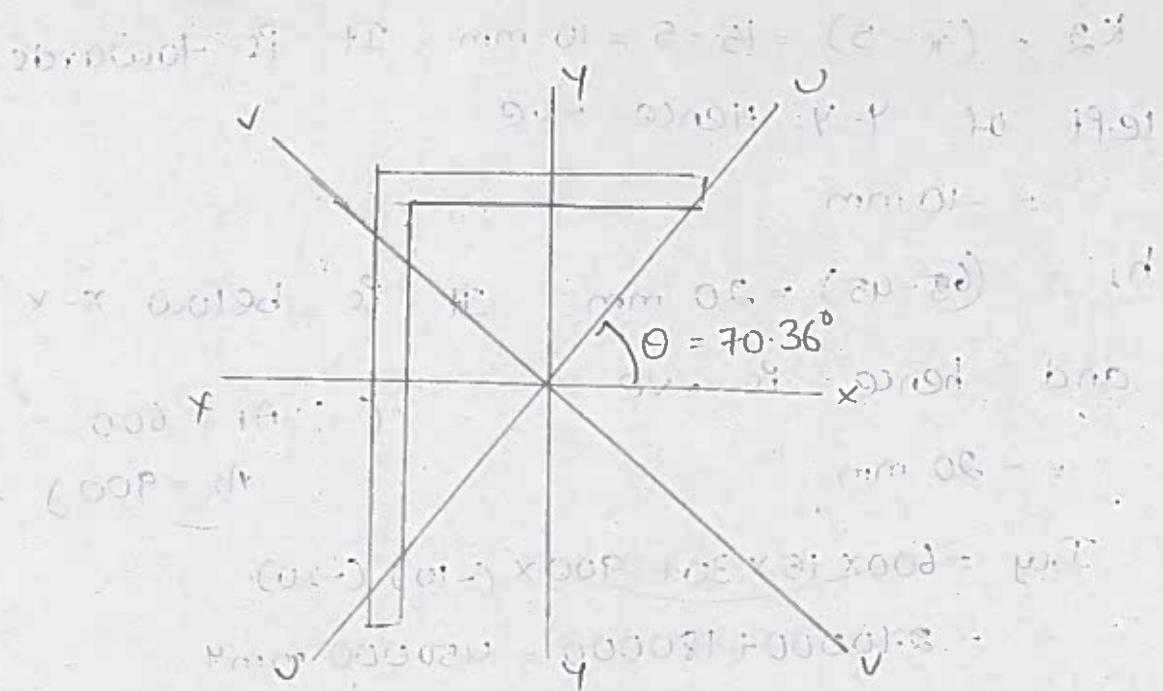
$$I_{OO} = \frac{I_{xx} + I_{yy}}{2} + \frac{(I_{xx} - I_{yy})}{2} \sec 2\theta$$

$$= \frac{1512500 + 412500}{2} + \frac{1512500 - 412500}{2} \sec 140.72^\circ$$

$$= 962500 + 550000 * \left[ -\frac{1}{0.744} \right] = 9$$

$$= 962500 - 710594$$

$$= 251905.6 \text{ mm}^4$$



$$\begin{aligned}
 I_{vv} &= I_{xx} + I_{yy} - I_{zz} \\
 &= 1512500 + 412500 - 251906 \\
 &= 1673094 \text{ mm}^4.
 \end{aligned}$$

$$\begin{aligned}
 \sigma_x &= \frac{P}{A} = \frac{1512500}{100 \times 50} = 30250 \text{ N/mm}^2 \\
 \sigma_y &= \frac{Q}{A} = \frac{412500}{100 \times 50} = 8250 \text{ N/mm}^2
 \end{aligned}$$